PAPE'S THEOREMS

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Geometry is a source of opportunity for students to explore mathematics on their own. Dan Pape, (IMSA Class of 1991), did a project in Geometry, which he subsequently expanded. Dan was intrigued by a problem; "find the length of the side of a regular hexagon whose area is numerically equal to its perimeter". Later he pursued the more general problem of finding the side of a regular n-gon whose area is numerically equal to its perimeter. $\left[\text{side} = 4\tan\left(\frac{180}{n}\right)\right]$. The interesting result was not, however, this formula. Dan observed a relationship that does not seem to be widely known.

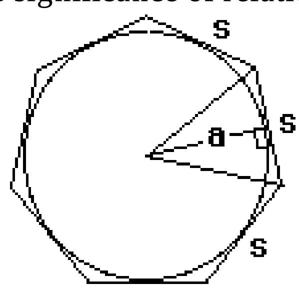
The area and perimeter of a regular polygon are equal whenever the radius of the inscribed circle is 2.

Since the proof is "trivial" Dan did not see the significance of relationship.

Perimeter = s•n and Area = $\frac{1}{2}$ a•p

s•n = $\frac{1}{2}$ a•p

s•n = $\frac{1}{2}$ a•s•n, so a = 2



s = length of side
a = apothem (in-radius)
n = number of sides

This is true regardless of the value of n.

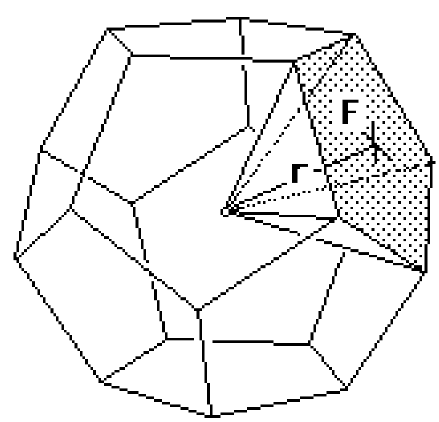
I told Dan that I had never noticed this "invariance" before. This stimulated him to extend the relationship to polyhedra:

The surface area of a regular polyhedron is equal to its volume whenever the radius of the inscribed sphere is **3**.

Volume = Surface Area

$\frac{1}{3}$ r •F•n = F•n

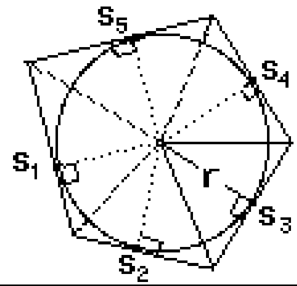
so: r = 3



This method of finding the volume further opened the door for a more general result:

Dan extended his results to **all** polygons that are circumscribed about a circle and to all polyhedra circumscribed about a sphere.

Circumscribed Polygon:



$$\text{Area} = \text{Perimeter}$$

$$\frac{1}{2} r s_1 + \frac{1}{2} r s_2 + \frac{1}{2} r s_3 + \ldots + \frac{1}{2} r s_n = s_1 + s_2 + s_3 + \ldots + s_n$$

$$\frac{1}{2} r (s_1 + s_2 + s_3 + \ldots + s_n) = s_1 + s_2 + s_3 + \ldots + s_n$$

$$r = 2$$

> If a polygon is circumscribed about a circle then the area is numerically equal to the perimeter if, and only if, the radius of the circle is 2.

Circumscribed Polyhedron:

$$\text{Volume} = \text{Surface Area}$$

$$\frac{1}{3} r F_1 + \frac{1}{3} r F_2 + \frac{1}{3} r F_3 + \ldots + \frac{1}{3} r F_n = F_1 + F_2 + F_3 + \ldots + F_n$$

$$\frac{1}{3} r (F_1 + F_2 + F_3 + \ldots + F_n) = F_1 + F_2 + F_3 + \ldots + F_n$$

$$r = 3$$

> If a polyhedron is circumscribed about a sphere then the volume is numerically equal to the surface area, if and only if, the radius of the sphere is 3.

What's Next? As a senior, Dan extended his conjecture to n-space.

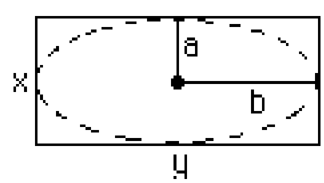
Excerpts taken from papers of Dan Pape, IMSA Alumni.

<u>Questions for further exploration</u>:

1. Under what conditions:
   a) Will the perimeter of a polygon circumscribed about a circle be numerically greater than the area?
   b) Will the surface area of a circumscribed polyhedron be numerically greater than the volume?

2. Is the area numerically equal to the perimeter for any other polygons besides those circumscribed about a circle? If so, under what conditions?

   Start with a simpler case:
   An x by y rectangle has "radii" a and b.

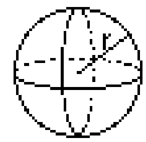
   Does there exist a rectangle (non-square) whose area and perimeter are numerically equal? What do we know about the set of all such rectangles?

For the Calculus Student

an interesting related phenomenon can be observed.

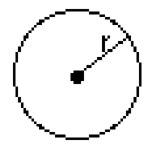
The volume of a sphere is: $V = \frac{4}{3}\pi r^3$

The derivative, $\frac{dV}{dr} = 4\pi r^2$ which is the surface area.



The area of a circle is: $A = \pi r^2$

The derivative, $\frac{dA}{dr} = 2\pi r$, which is the circumference. (perimeter)



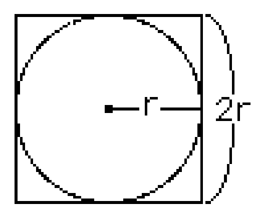
What would happen with a cube ?

The volume of a cube is: $V = s^3$, but the derivative is $\frac{dV}{ds} = 3s^2$, which is NOT the surface area.

The area of a square is $A = s^2$, but the derivative is $\frac{dA}{ds} = 2s$. NOT the perimeter.

If we reflect back to the previous article and use the same perspective: Let's look at the apothem of the cube and square. The apothem is the radius of the inscribed sphere or circle and thus resembles the circle / sphere example.

Write the perimeter, area and volume in terms of the in-radius.

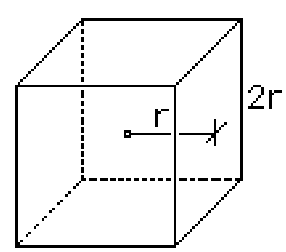


Area = $4r^2$. The derivative: $\frac{dA}{dr} = 8r$, which is the perimeter.

We must just take the derivative with respect to the proper variable, **r**.

The cube problem now becomes:

Volume = $(2r)^3 = 8r^3$ So the derivative is $\frac{dV}{dr} = 24r^2$, which is the surface area.



If you use the appropriate variable; In a solid, the derivative of volume is surface area and in the plane, the derivative of area is perimeter.

See what other connections you can find !