EXERCISING WITH THE MATHEMATICAL INDUCTION PRINCIPLE

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In my Analysis IV classes, as an application of the Mathematical Induction Principle, the students were invited to find patterns for the sums

$$S_n(m) = 1^m + 2^m + \ldots + n^m,$$

$m = 1, 2, 3$, and to validate them by mathematical induction.

While doing this they discovered the identity $S_n(3) = \left(S_n(1)\right)^2$.

One student truly liked this relation:

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2 \quad (*)$$

and asked me if there were any other positive integer sequences $\left(a_n\right)$ which satisfy a corresponding relation.

It turned out that the only sequence of positive numbers $a_1, a_2, \ldots, a_n, \ldots$ (not necessarily integer!) for which the equality

$$a_1^3 + a_2^3 + \ldots + a_n^3 = \left(a_1 + a_2 + \ldots + a_n\right)^2 \quad (**)$$

held for every positive integer n was the original sequence, given by $a_n = n$.

To justify this, we will use the Mathematical Induction Principle (**strong form**):

If P(n) is a statement concerning the natural number n, and n_0 is a fixed non-negative integer, to prove that P(n) is true for all $n \geq n_0$ it is sufficient

to show that: (i) $P\left(n_0\right)$ *is true.*

(ii) *For every $k \geq n_0$, if $P(n_0 + 1)$, $P(n_0 + 2)$, . . . , P(k) are true, then P(k + 1) is true.*

Let P(n) be $a_n = n$, $n \geq 1$. P(1) is evident because $a_1^3 = a_1^2$ and $a_1 \neq 0$ lead to $a_1 = 1$.

Assuming $P(l)$ is true for $2 \leq l \leq k$, we have $a_1 = 1$, $a_2 = 2$, . . . , $a_k = k$ and using (**), which is valid for every **n** (hence true for n = k + 1), we get :

$$1^3 + 2^3 + \ldots + k^3 + a_{k+1}^3 = \left(1 + 2 + \ldots + k + a_{k+1}\right)^2.$$

Expanding the right hand side by the formula $(A + B)^2 = A^2 + 2AB + B^2$, where $A = 1 + 2 + \dots + k$, $B = a_{k+1}$ and cancelling out $1^3 + 2^3 + \dots + k^3$ and $(1 + 2 + \dots + k)^2$, we obtain : $a_{k+1}^3 = 2(1 + 2 + \dots + k)\, a_{k+1} + a_{k+1}^2$.

Replacing $1 + 2 + \dots + k$ by $\frac{k(k+1)}{2}$ and dividing by $a_{k+1} \neq 0$ we acquire

$$a_{k+1}^2 = k(k + 1) + a_{k+1} ,$$

which is a quadratic in a_{k+1}. This equation has $-k$ and $k + 1$ as solutions. Since a_{k+1} is positive we deduce that $a_{k+1} = k + 1$, therefore, $P(k + 1)$ is also true and, by the Mathematical Induction Principle, $a_n = n$ for every positive integer **n**.

I discussed this result with the mathletes in one of my Advanced Problem Solving sessions and I challenged them to prove, more generally, that

If $a_1, a_2, \dots, a_n$ are distinct positive integers, then

$$a_1^3 + a_2^3 + \dots + a_n^3 \geq \left(a_1 + a_2 + \dots + a_n\right)^2 .$$

It seemed difficult for them to prove this inequality but they came up with some good ideas. The following solution is also based on the Mathematical Induction Principle (**weak form**):

If $P(n)$ is an assertion about the natural number n, and n_0 is a fixed non-negative integer, to prove that $P(n)$ is true for all $n \geq n_0$ it suffices to show that:

(i) $P(n)$ is true for $n = n_0$..

(ii) For every $k \geq n_0$, if $P(k)$ is true , then $P(k + 1)$ is true.

One can assume (without loss of generality) that $a_1 < a_2 < \dots < a_n$, so this will become part of the hypothesis. For $n = 1$ the inequality is obvious, because $a_1^3 \geq a_1^2$ for $a_1 \geq 1$.

Let's suppose it is true for $n = k$, i.e. $a_1^3 + a_2^3 + \dots a_k^3 \geq \left(a_1 + a_2 + \dots + a_k\right)^2$ (1)

and according to our presumption $\left(a_1 < a_2 < \dots\right)$ let a_{k+1} be an integer greater than a_k.

Then $a_{k+1} - 1 \ge a_k$ (◊) so $a_{k+1} \ge a_k + 1$, therefore;

$$\frac{\left(a_{k+1} - 1\right)a_{k+1}}{2} \ge \frac{a_k\left(a_k + 1\right)}{2} = 1 + 2 + \dots + a_k \overset{(\lozenge\lozenge)}{\ge} a_1 + a_2 + \dots + a_k\,,$$

whence, multiplying by $2 \cdot a_{k+1}$, we obtain:

$$\left(a_{k+1}^2 - a_{k+1}\right)a_{k+1} \ge 2\left(a_1 + a_2 + \dots + a_k\right)a_{k+1}$$

i.e. $$a_{k+1}^3 \ge 2\left(a_1 + a_2 + \dots + a_k\right)a_{k+1} + a_{k+1}^2 \qquad (2)$$

Adding up (1) and (2) we get:

$$a_1^3 + a_2^3 + \dots + a_k^3 + a_{k+1}^3 \ge \left(a_1 + a_2 + \dots + a_k + a_{k+1}\right)^2,$$

so our inequality is true for n = k + 1, and our inductive proof is completed.

From (◊) and (◊◊) we infer that the equality holds, if and only if, for every positive integer k the equalities $a_{k+1} - 1 = a_k$ and $\left\{1, 2, \dots, a_k\right\} = \left\{a_1, a_2, \dots, a_k\right\}$ occur i.e. $a_k = k$, k = 1, 2, . . . n , the same conclusion as in the first problem. ✍