Calculator Curiosities: Order of Operations, Domain, and Inverse Functions

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The calculator has changed the way in which we learn and do mathematics. It also changes the way we think. Calculators can give us new insights into mathematical ideas and allow us to see things that otherwise remain hidden. Calculators also present us with many curiosities that arise due to ideosyncracies ("undocumented features") in their design.

Try each of the following on a graphing calculator:
Enter each of them exactly as shown, including √, ( ), * [times] and ^ [exponent].
Are the results correct? Are they what you expected? How does *order of operations* play a role in the results?

1. Calculate: a. $1 \div 2 + 3$ c. $2 + 3^{-1}$ e. $(-8)^{(1/3)}$ g. $(-8)^{(5/3)}$
   b. $1 \div (2 + 3)$ d. -2^4 f. $(-8)^{(5)}$ h. $(-8)^{(6/2)}$

2. Graph: a. y = [√] 2 x c. y = [√] x + 4
   b. y = [√] 2 * xd. y = [√] (x + 4)

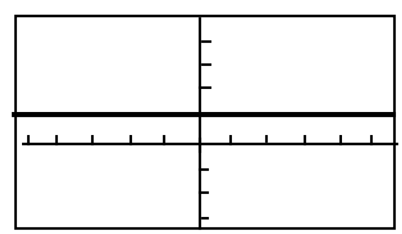
3. Graph: a. y = 3 ^ 2 x c. y = (3 ^ 2) * x
   b. y = 3 ^ 2 * x d. y = 3 ^ (2x)

4. Graph: a. y = cos 2 * xc. y = 9 [/] 2 x
   b. y = cos 2 x d. y = 9 [/] 2 * x

5. Graph: a. y = log 2 x c. y = log (2 x)
   b. y = log 2 * xd. y = (log 2) x

6. Graph: $y_1 = \log x^2$ and $y_2 = 2 \log x$. Are the graphs the same?
   Should they be the same? Explain what happens.

7. Paul tried to graph: y = cos x using the standard zoom on a TI-81 calculator. He was surprised to obtain the graph shown at the right. Paul knew that this horizontal line was not the cosine graph that he expected. What did he do wrong? Try it on your calculator.



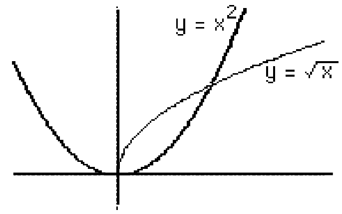
Calculators can help you to see things in a different way. They can also force you to clarify your understanding of mathematical concepts. Take for example the concept of inverse function:

The inverse, $\mathbf{f}^{-1}$, of the function, **f,** is defined such that the following equalities:

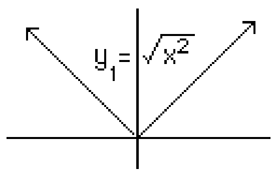
$\mathbf{f}^{-1}(\mathbf{f}(x)) = \mathbf{f}(\mathbf{f}^{-1}(\mathbf{x})) = x$ are satisfied for every suitable x .

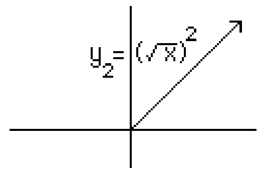
Using a graphing calculator we should be able to verify this fact.
That is, we should be able to graph: $y = \mathbf{f}^{-1}(\mathbf{f}(x))$ and $y = \mathbf{f}(\mathbf{f}^{-1}(\mathbf{x}))$. Each of the graphs should be the same subset of the graph of $\mathbf{y} = \mathbf{x}$, which is a straight line at 45° to the positive x-axis.

Ex: 1. Graph: $y = x^2$ and $y = \sqrt{x}$, to recall what the original functions look like.



Now, let's graph:
$y_1 = \sqrt{x^2}$ and $y_2 = (\sqrt{x})^2$



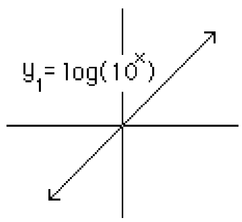


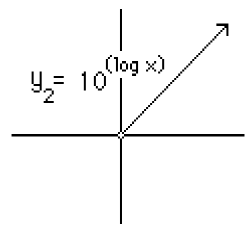
Why are the graphs different? One graph is a subset of y = x and the other is not. Why?

Recall that the "suitable" domain for which $y = x^2$ has an inverse is the restricted domain of non-negative real numbers. The graphs are identical for $x \geq 0$.

Notice that the graphs of $y_1 = \sqrt{x^2}$ and $y_2 = |\ x\ |$ are the same, so $\sqrt{x^2} = |x|$.

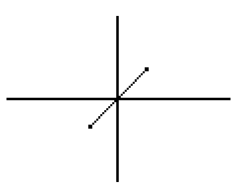
Ex: 2. Now we shall try to graph: $y_1 = \log(10^x)$ and $y_2 = 10^{(\log x)}$

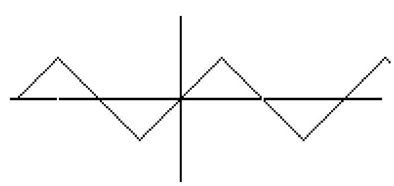




Observe that both graphs are subsets of y = x, but they are not the same.
Here again we find that the inverse has a restricted domain: $x > 0$.
Explain why some graphs include more than the restricted domain allows.

- See if you can explain the result when we obtain the following graphs:
  $y_1 = \sin(\sin^{-1}(x))$ and $y_2 = \sin^{-1}(\sin(x))$





  Try to determine the endpoints of the segments in these graphs.

- See if you can predict the graphs of: $y = \mathbf{f}^{-1}(\mathbf{f}(x))$ and $y = \mathbf{f}(\mathbf{f}^{-1}(\mathbf{x}))$ for the following functions. Then try the graphs on your calculator.
  a) $f(x) = \cos(x)$ b) $f(x) = \tan(x)$ c) $f(x) = x^3$ d) $f(x) = 1/x$. ✍